1 (a) We want to solve

$$y' - 2y = 1$$

on $I = (-\infty, \infty)$.

Let $A(x) = \int -2\,dx = -2x$

Multiply by $e^{A(x)} = e^{-2x}$ to get

$$e^{-2x}y' - 2e^{-2x}y = e^{-2x}$$

This gives

$$\underbrace{(e^{-2x}y)'}_{\text{this is always } (e^{A(x)}y)'} = e^{-2x}$$

Integrating with respect to x gives

$$e^{-2x}y = \int e^{-2x}\,dx$$

So, $e^{-2x} \cdot y = -\frac{1}{2}e^{-2x} + C$

Then, $y = \dfrac{-\frac{1}{2}\cancel{e^{-2x}}}{\cancel{e^{-2x}}} + \dfrac{C}{e^{-2x}}$

$$\frac{1}{e^{-2x}} = e^{2x}$$

Answer: $\boxed{y = -\frac{1}{2} + Ce^{2x}}$

1(b)

We want to solve

$$y' + 2xy = x$$

on $I = (-\infty, \infty)$

Let $A(x) = \int 2x\,dx = x^2$

Multiply by $e^{A(x)} = e^{x^2}$ to get

$$e^{x^2} y' + 2x e^{x^2} y = x e^{x^2}$$

this is always $\left(e^{A(x)} y\right)'$

So, $$\left(e^{x^2} \cdot y\right)' = x e^{x^2}$$

Thus, by integrating with respect to x we get

$$e^{x^2} \cdot y = \int x e^{x^2}\,dx$$

We have that

$$\int x e^{x^2}\,dx = \int \tfrac{1}{2} e^{u}\,du = \tfrac{1}{2} e^{u} + C = \tfrac{1}{2} e^{x^2} + C$$

$u = x^2$
$du = 2x\,dx$
$\tfrac{1}{2}\,du = x\,dx$

Thus,

$$e^{x^2} \cdot y = \tfrac{1}{2} e^{x^2} + C$$

So,

$$y = \frac{\frac{1}{2}\cancel{e^{x^2}}}{\cancel{e^{x^2}}} + \frac{c}{e^{x^2}}$$

Answer: $\boxed{y = \frac{1}{2} + C e^{-x^2}}$

1 (c) We want to solve

$$\frac{dy}{dx} + 2xy = xe^{-x^2}$$

on $I = (-\infty, \infty)$

Let $A(x) = \int 2x\,dx = x^2$

Multiply both sides by $e^{A(x)} = e^{x^2}$ to get

$$e^{x^2} \cdot \frac{dy}{dx} + e^{x^2} \cdot 2xy = e^{x^2} x e^{-x^2}$$

$$e^{x^2} e^{-x^2} = e^{x^2 - x^2} = e^0 = 1$$

or

$$e^{x^2} e^{-x^2} = \frac{e^{x^2}}{e^{x^2}} = 1$$

Thus,

$$\left(e^{x^2} \cdot y\right)' = x$$

this is always

$$\left(e^{A(x)} y\right)'$$

Integrating with respect to x gives:

$$e^{x^2} \cdot y = \int x\,dx$$

$$e^{x^2} \cdot y = \frac{x^2}{2} + C$$

Thus, $y = \dfrac{x^2/2}{e^{x^2}} + \dfrac{C}{e^{x^2}}$

Answer $\boxed{y = \frac{1}{2}x^2 e^{-x^2} + Ce^{-x^2}}$

$\boxed{①(d)}$ We want to solve

$$y' + \frac{1}{x}y = 3x^2 - \frac{1}{x}$$

on $I = (0, \infty)$.

Let

$$A(x) = \int \frac{1}{x}\,dx = \ln|x| = \ln(x)$$

$x > 0$
on $I = (0, \infty)$
so, $|x| = x$

We get

$$e^{A(x)} = e^{\ln(x)} = x$$

Recall
$$e^{\ln(B)} = B$$

multiply both sides
of $y' + \frac{1}{x}y = 3x^2 - \frac{1}{x}$
by x to get:

$$xy' + y = 3x^3 - 1$$

This becomes

$$\underbrace{(xy)'}_{} = 3x^3 - 1$$

this is always
$$\left(e^{A(x)}y\right)'$$

Thus, by integrating both sides with respect to x we get

$$xy = \int (3x^3 - 1)\,dx$$

So,

$$xy = \frac{3}{4}x^4 - x + C$$

Thus,

$$\boxed{y = \frac{3}{4}x^3 - 1 + \frac{C}{x}}$$

Answer

2 (a) We want to solve

$$3\frac{dy}{dx} + y = 2e^{-x}$$

on $I = (-\infty, \infty)$.

Divide by 3 to put the ODE into standardized form. We get

$$\frac{dy}{dx} + \frac{1}{3}y = \frac{2}{3}e^{-x}$$

Let $A(x) = \int \frac{1}{3}dx = \frac{1}{3}x$.

Multiply both sides by $e^{A(x)} = e^{\frac{1}{3}x}$ to get

$$e^{\frac{1}{3}x} \cdot \frac{dy}{dx} + \frac{1}{3}e^{\frac{1}{3}x}y = \frac{2}{3}\underbrace{e^{\frac{1}{3}x}e^{-x}}$$

$$e^{\frac{1}{3}x} \cdot e^{-x} = e^{\frac{1}{3}x - x} = e^{-\frac{2}{3}x}$$

So,

$$\underbrace{\left(e^{\frac{1}{3}x} \cdot y\right)'} = \frac{2}{3}e^{-\frac{2}{3}x}$$

this is always $\left(e^{A(x)}y\right)'$

Thus,

$$e^{\frac{1}{3}x} \cdot y = \underbrace{\int \frac{2}{3}e^{-\frac{2}{3}x}dx}$$

$$\int \frac{2}{3}e^{-\frac{2}{3}x}dx = \frac{2}{3} \cdot \left(-\frac{3}{2}e^{-\frac{2}{3}x}\right) + C$$

$$= -e^{(-2/3)x} + C$$

Thus,

$$e^{\frac{1}{3}x} \cdot y = -e^{-\frac{2}{3}x} + c$$

So,

$$y = \frac{-e^{-\frac{2}{3}x}}{e^{\frac{1}{3}x}} + \frac{c}{e^{\frac{1}{3}x}}$$

$$y = -e^{-\frac{2}{3}x - \frac{1}{3}x} + Ce^{-\frac{1}{3}x}$$

$$\boxed{y = -e^{-x} + Ce^{-\frac{1}{3}x}}$$

Answer

$\boxed{②(b)}$ We want to solve

$$x^2y' + x(x+2)y = e^x$$

on $I = (0,\infty)$

First divide by x^2 to put the ODE into standardized form. We get

$$y' + \left(1+\frac{2}{x}\right)y = x^{-2}e^x$$

Let

$$A(x) = \int \left(1+\frac{2}{x}\right)dx = x + 2\ln|x| = x + 2\ln(x)$$

since $I = (0,\infty)$ we have $x > 0$ so $|x| = x$

We have

$$e^{A(x)} = e^{x+2\ln(x)} = e^x e^{2\ln(x)} = e^x e^{\ln(x^2)} = x^2e^x$$

So multiply both sides of $y' + \left(1+\frac{2}{x}\right)y = x^{-2}e^x$ by x^2e^x to get:

$$x^2e^xy' + x^2e^x\left(1+\frac{2}{x}\right)y = x^2e^xx^{-2}e^x$$

This simplifies to

$$x^2e^xy' + (x^2+2x)e^xy = e^{2x}$$

We get

$$\underbrace{(x^2 e^x y)'}_{} = e^{2x}$$

this is always $(e^{A(x)} y)'$

Integrating with respect to x gives

$$x^2 e^x y = \frac{1}{2} e^{2x} + C$$

So,

$$y = \frac{\frac{1}{2} e^{2x}}{x^2 e^x} + \frac{C}{x^2 e^x}$$

$$y = \frac{1}{2} x^{-2} e^{2x-x} + C x^{-2} e^{-x}$$

$$\boxed{y = \frac{1}{2} x^{-2} e^{x} + C x^{-2} e^{-x}}$$

Answer

$\boxed{②(c)}$ We want to solve

$$(x^2+9)\frac{dy}{dx} + xy = 0$$

on $I = (-\infty, \infty)$

Divide by x^2+9 to put the ODE into a standardized form. We get

$$\frac{dy}{dx} + \frac{x}{x^2+9}\, y = 0$$

Let $A(x) = \int \frac{x}{x^2+9}\, dx = \int \frac{1}{2}\frac{1}{u}\, du = \frac{1}{2}\ln|u|$

$= \frac{1}{2}\ln|x^2+9|$

$= \frac{1}{2}\ln(x^2+9)$

$u = x^2+9$
$du = 2x\,dx$
$\frac{1}{2}du = x\,dx$

$x^2+9>0$ always

Multiply both sides of the ODE by

$$e^{A(x)} = e^{\frac{1}{2}\ln(x^2+9)} = e^{\ln((x^2+9)^{1/2})} = (x^2+9)^{1/2}$$

to get

$$(x^2+9)^{1/2}\cdot\frac{dy}{dx} + (x^2+9)^{1/2}\frac{x}{(x^2+9)}\, y = 0$$

This simplifies to

$$(x^2+9)^{1/2}\frac{dy}{dx}+\frac{x}{(x^2+9)^{1/2}}\,y=0$$

This becomes

$$\underbrace{\left[(x^2+9)^{1/2}\cdot y\right]'}=0$$

this is always

$$\left(e^{A(x)}y\right)'$$

Integrating with respect to x gives

$$(x^2+9)^{1/2}\cdot y = C$$

So,

$$\boxed{y=\frac{C}{(x^2+9)^{1/2}}=\frac{C}{\sqrt{x^2+9}}}$$

3 We want to solve

$$\frac{dy}{dx} + 2xy = x, \quad y(0) = -3$$

on $I = (-\infty, \infty)$

First we must find the general solution to

$$\frac{dy}{dx} + 2xy = x$$

Let

$$A(x) = \int 2x\,dx = x^2$$

multiply both sides by $e^{A(x)} = e^{x^2}$ to get

$$e^{x^2} \cdot \frac{dy}{dx} + 2xe^{x^2}y = xe^{x^2}$$

this is always $(e^{A(x)}y)'$

This gives

$$\underbrace{(e^{x^2} \cdot y)'}_{} = xe^{x^2}$$

Integrating both sides with respect to x gives

$$e^{x^2} \cdot y = \int xe^{x^2}\,dx$$

$$\int xe^{x^2}\,dx = \int \frac{1}{2}e^u\,du = \frac{1}{2}e^u = \frac{1}{2}e^{x^2} + C$$

$u = x^2$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

So,

$$e^{x^2}\cdot y = \frac{1}{2}e^{x^2} + C$$

Thus,

$$y = \frac{1}{2}\underbrace{e^{-x^2}\cdot e^{x^2}} + Ce^{-x^2}$$

$$\boxed{e^{-x^2}\cdot e^{x^2} = e^{-x^2+x^2} = e^0 = 1}$$

So,

$$y = \frac{1}{2} + Ce^{-x^2}$$

We want $y(0) = -3$. Plugging this into the above we get

$$-3 = y(0) = \frac{1}{2} + Ce^{-(0)^2}$$

So,

$$-3 = \frac{1}{2} + Ce^0 = \frac{1}{2} + C$$

Thus,

$$C = -3 - \frac{1}{2} = -\frac{7}{2}.$$

So,

$$\boxed{y = \frac{1}{2} - \frac{7}{2}e^{-x^2}}$$

4 We want to solve

$$xy' + y = 2x, \quad y(1) = 0$$

on $I = (0, \infty)$

First put the equation into a standardized form by dividing through by x to get

$$y' + \frac{1}{x} y = 2$$

Let

$$A(x) = \int \frac{1}{x}\,dx = \ln|x| = \ln(x)$$

$x > 0$ since $I = (0, \infty)$

multiply both sides by

$$e^{A(x)} = e^{\ln(x)} = x$$

to get

$$xy' + y = 2x$$

This gives

$$\underbrace{(x \cdot y)'}_{\text{always } (e^{A(x)} y)'} = 2x$$

Integrating with respect to x gives

$$x \cdot y = \int 2x\,dx$$

So,

$$x \cdot y = x^2 + c$$

Thus,

$$y = x + \frac{c}{x}$$

We want $y(1) = 0$. Plugging this in gives

$$0 = y(1) = 1 + \frac{c}{1}$$

So,

$$0 = 1 + c$$

Thus,

$$c = -1.$$

Therefore, the solution is

$$\boxed{y = x - \frac{1}{x}}$$

5. We saw in the previous problems that the general solution to

$$(x^2+9)\frac{dy}{dx} + xy = 0$$

on $I = (-\infty, \infty)$ is

$$y = \frac{c}{\sqrt{x^2+9}}$$

We want the solution to also satisfy $y(0)=1$. plug in $x=0$ to get

$$1 = y(0) = \frac{c}{\sqrt{0^2+9}}$$

So, $1 = \frac{c}{3}$.

Thus, $c = 3$.

So the solution is

$$y = \frac{3}{\sqrt{x^2+9}}$$